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LETTER TO THE EDITOR

Equivalence of $(d+1)$-dimensional Ising systems of arbitrary spin to a d-dimensional spin-$\frac{1}{2}$ quantum system

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Abstract. In a certain highly anisotropic limit the spin-S Ising model in $d+1$ dimensions is shown to be equivalent to the d-dimensional spin-$\frac{1}{2}$ Ising model in a transverse field at zero temperature. This result relates, in the highly anisotropic limit, a group of models in the same universality class.

In a special limit of extreme lattice anisotropy, the $(d+1)$-dimensional spin-$\frac{1}{2}$ Ising model is identical (Pfeuty 1976, Suzuki 1976, Fradkin and Susskind 1978) to the zero temperature limit of the d-dimensional spin-$\frac{1}{2}$ Ising model in a transverse field (TIM (d)). The latter system is quantum mechanical with the (continuous) imaginary time being identified with the extra dimension of the $(d+1)$-dimensional system.

The purpose of this Letter is to point out that universality then suggests† the equivalence of the near critical behaviour of the TIM (d) and that of $(d+1)$-dimensional Ising models of arbitrary anisotropy (including the isotropic case) and *arbitrary spin*; and to give an explicit connection between $(d+1)$-dimensional spin-S Ising models (IM $(d+1, S)$) in an anisotropic limit and the TIM (d). This connection is not restricted to the critical regime.

The anisotropic limit is such that the IM $(d+1, S)$ consists of n weakly coupled chains: d-dimensional layers have weak coupling K_w in the layers and strong coupling K_s between the layers. The limit $K_s \to \infty$, $K_w \to 0$ is taken with any fixed λ where

$$\lambda = 2S(K_w\, e^{2K_s})^{-1}. \qquad (1)$$

The equivalent TIM (d) has $\Gamma/J = \lambda$ where Γ is the transverse field and J is the exchange interaction.

Since the connection is for all S, we thus have an equivalence between the highly anisotropic limits of a group of models in the same universality class (the IM $(d+1, S)\ \forall S$). This equivalence was also obtained by Sneddon and Stinchcombe (1978) and is complemented by the results of Stoeckly and Scalapino (1975). They show that, in the anisotropic limit, the ϕ^4 field theory, (which is in the same universality class as the Ising models (Wilson and Kogut 1974) also reduces to the spin-$\frac{1}{2}$ transverse Ising model in one lower dimension.

† This assumes that the special anisotropic limit does not change the critical behaviour. This limit is discussed by Sneddon and Stinchcombe (to be published).

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Analogous results also obtain for other classes of models. For example, the X–Y (plane rotor) model and the periodic Gaussian model (Villain 1975) are in the same universality class. In their appropriate highly anisotropic limits these models become identical (Green 1978). They are then both described by the Hamiltonian for coupled quantum rotors in one lower dimension. In this case the scalings appropriate to the anisotropic limits are $K_s \to \infty$, $K_w \to 0$ and fixed λ where, for the X–Y model

$$\lambda = K_s K_w$$

and for the periodic Gaussian (Villain) model

$$\lambda = 2K_s \, e^{-1/2K_w}.$$

These results generalise to other models of the Villain type (Green 1978).

The connection to be established in this letter could be arrived at by combining the results of Fradkin and Susskind (1978) and Sneddon and Stinchcombe (1978). These respectively relate the Hamiltonian of the TIM (d) and the transfer matrix of the anisotropic IM ($d+1, S$)‡ to the transfer matrix of the anisotropic IM ($d+1, \frac{1}{2}$). In what follows, however, the connection will be established directly, by generalising an argument of Fradkin and Susskind (1978) and using some single chain properties derived by Sneddon and Stinchcombe (1978).

The reduced Hamiltonian of the spin-$\frac{1}{2}$ transverse Ising model can be written

$$-\beta \mathcal{H}(J, \Gamma) = \gamma \boldsymbol{H}\left(\frac{J}{\Gamma}\right)$$

where H is a one-parameter operator given by

$$\boldsymbol{H}\left(\frac{J}{\Gamma}\right) = \left(\frac{J}{\Gamma}\right) \sum_{\langle ij \rangle} \boldsymbol{\sigma}_i^x \boldsymbol{\sigma}_j^x + \sum_i \boldsymbol{\sigma}_i^z \tag{2}$$

$K = \beta J$, $\gamma = \beta\Gamma$; $\Sigma_{\langle ij \rangle}$ is a sum over nearest neighbour sites in a d-dimensional lattice; and $\boldsymbol{\sigma}^x$, $\boldsymbol{\sigma}^z$ are the usual Pauli matrices.

The transfer matrix between two adjacent d-dimensional layers of n spin-S Ising spins, with couplings as defined above, can be written

$$T_{\alpha\beta} = \exp\left[K_s\left(S^{-2} \sum_i \sigma_i(\alpha)\sigma_i(\beta) - n\right) + K_w S^{-2} \sum_{\langle ij \rangle} \sigma_i(\alpha)\sigma_j(\alpha)\right]. \tag{3}$$

Here α and β label respectively the states of the two layers; the σ_i take the values $-S$, $-S+1, \ldots, S-1, S$; and $\Sigma_{\langle ij \rangle}$ is a sum over nearest neighbour pairs in one layer. Expanding (3), in the limit defined by (1), and retaining only terms to $O(e^{-2K_s})$ gives

$$\boldsymbol{T}(K_w, K_s) = \boldsymbol{T}_1(K_s) + K_w \boldsymbol{T}_2$$

where $\boldsymbol{T}_1(K_s)$ is the transfer matrix of n uncoupled spin-S Ising chains and $\boldsymbol{T}_2 =$

‡ Sneddon and Stinchcombe (1978) are concerned with the case of $d+1=2$, but the generalisation to all d of the particular result referred to here is immediate.

$\Sigma_{\langle ij\rangle} \tau_i^z \tau_j^z$ where τ^z is a rank $(2S+1)$ matrix given by

$$\tau^z = \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & -1 \end{pmatrix}.$$

A direct product of single chain matrices $\boldsymbol{a}$ can be used to diagonalise $\boldsymbol{T}_1$. The leading order form of the single chain matrix $\boldsymbol{a}$ satisfies (Sneddon and Stinchcombe 1978)

$$\begin{matrix} 2^{-1/2} = a_{1,s} = \pm a_{1,-s} \\ 2^{-1/2} = a_{2,s} = \mp a_{2,-s} \end{matrix} \qquad a_{j,\sigma} = 0 \qquad \text{if} \begin{cases} j = 1, 2; \sigma \neq \pm s \\ j \neq 1, 2; \sigma = \pm s \end{cases}.$$

To leading order, the τ^z matrices thus transform to

$$\tau^x = \left(\begin{array}{cc:c} 0 & 1 & \\ 1 & 0 & \\ \hdashline & & \mathbf{0} \end{array}\right).$$

In this basis, $\boldsymbol{T}_1$ is a product of diagonal single chain operators whose eigenvalues satisfy (Sneddon and Stinchcombe, 1978)

$$\lambda_1, \lambda_2 \sim 1, \qquad \lambda_1 - \lambda_2 = 4S\, \mathrm{e}^{-2K_s}(1+\epsilon), \qquad \text{where } \epsilon \to 0 \text{ as } K_s \to \infty \tag{4}$$
$$\lambda_j \sim \mathrm{e}^{-qK_s} \qquad \text{where } q > 0 \text{ for } j > 2.$$

Thus, in the new basis, the problem decouples into one part, formed from the single chain eigenstates with eigenvalues λ_1 and λ_2, and a second part formed from the other single chain eigenstates. Only the first part, $\boldsymbol{T}'$ say, has eigenvalues which do not vanish as $K_s \to \infty$. In the new basis then, to $\mathrm{O}(\mathrm{e}^{-2K_s})$,

$$\boldsymbol{T}' = \prod_i \boldsymbol{d}'_i + K_w \sum_{\langle ij\rangle} \boldsymbol{\sigma}_i^x \boldsymbol{\sigma}_j^x$$

where

$$\boldsymbol{d}'_i = \begin{pmatrix} \lambda_1(S, K_s) & 0 \\ 0 & \lambda_2(S, K_s) \end{pmatrix}.$$

Using (4) and (2) then gives, to $\mathrm{O}(\mathrm{e}^{-2K_s})$,

$$\boldsymbol{T}' = \lambda_1^n(S, K_s)\boldsymbol{I} + 2S\, \mathrm{e}^{-2K_s}[\boldsymbol{H}(K_w\, \mathrm{e}^{2K_s}/2S) - n\boldsymbol{I}].$$

Thus the Hamiltonians giving the thermodynamics and correlation functions of the anisotropic Ising models, for any S, are related to the Hamiltonian giving the zero temperature dynamics and correlation functions of the quantum model. In particular the time evolution given by the Hamiltonian (2) is equivalent to the spatial decay of correlations in the strongly coupled direction of the classical model, in the limit given by (1), for any value of λ. Another consequence of this relation is that the criticality of the d-dimensional transverse Ising model at a particular value $\lambda_c(d)$ of Γ/J gives, using (1), the leading order form of the critical curve for the anisotropic $(d+1)$-dimensional spin-S Ising model. This result was given previously (Sneddon and Stinchcombe 1978) for the special case of $d+1=2$.

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References

Fradkin E and Susskind L 1978 *Phys. Rev.* D **17** 2637
Green M B 1978 *Nucl. Phys.* **144** B 473
Pfeuty P 1976 *J. Phys. C: Solid St. Phys.* **9** 3993
Sneddon L and Stinchcombe R B 1978 *J. Phys. C: Solid St. Phys.* **11** 4037
Stoeckly B and Scalapino D J 1975 *Phys. Rev.* B **11** 205
Suzuki M 1976 *Prog. Theor. Phys.* **56** 1454
Villain J 1975 *J. Physique* **36** 583
Wilson K and Kogut J 1974 *Phys. Rep.* **12** C 75